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## Mathematics for Computer Scientists 1, WS 2017/18
## Sheet 11

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**1.** This exercise is to be solved using the intermediate-value theorem.

(i) Let $\alpha < \beta$. Show that the equation

$$\frac{x^2+1}{x-\alpha} + \frac{x^6+1}{x-\beta} = 0$$

has at least one solution $x_0 \in (\alpha, \beta)$.

(ii) Show that the equation

$$2^x = 4x$$

has at least one solution other than $x = 4$.

**2.** (a) Sketch the graph of the function $f : \mathbb{R} \to \mathbb{R}$ mit

$$f(x) = \frac{1}{1+|x|} + \frac{1}{1+|x-a|},$$

where $a$ is a positive constant, and show that the maximum value of $f$ is

$$\frac{2+a}{1+a}.$$

(b) Let $p$ be a polynomial of degree $n$ with critical points $-1$, $1$, $2$, $3$ and $4$. The corresponding values of $p$ are $6$, $1$, $2$, $4$ and $3$ and the coefficient of $x^n$ is $1$. Sketch the graph of $p$, distinguishing between the cases $n$ even and $n$ odd.

**3.** This exercise is to be solved using the following result which is proved in lectures.

Consider the continuous, bijective function $f : I \to J$ with continuous inverse $f^{-1} : J \to I$, where $I$, $J$ are open intervals.

$f^{-1}$ is differentiable at the point $b$ if and only if $f$ is differentiable at the point $a = f^{-1}(b)$ and $f'(a) \neq 0$. In this case

$$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

(a) Let $n$ be a natural number. Define

$$g_n(x) = x^{\frac{1}{n}}, \qquad x \in \mathbb{R},$$

if $n$ is odd and

$$g_n(x) = x^{\frac{1}{n}}, \qquad x \in [0, \infty),$$

if $n$ is even.

(i) Show that $g_n$ is differentiable for $x \neq 0$ and $g_n'(x) = \frac{1}{n} x^{\frac{1}{n}-1}$.

(ii) Deduce that

$$\frac{\mathrm{d}}{\mathrm{d}x} x^q = q x^{q-1}$$

for all *positive* rational numbers $q$.

(iii) Deduce that

$$\frac{\mathrm{d}}{\mathrm{d}x} x^q = q x^{q-1}$$

for all *negative* rational numbers $q$.

(b) The inverses of

$$\sin(\cdot) : [-\pi/2, \pi/2] \to [-1, 1], \qquad \cos(\cdot) : [0, \pi] \to [-1, 1],$$

$$\tan(\cdot) : (-\pi/2, \pi/2) \to \mathbb{R}$$

are denoted by

$$\arcsin(\cdot) : [-1, 1] \to [-\pi/2, \pi/2], \qquad \arccos(\cdot) : [-1, 1] \to [0, \pi],$$

$$\arctan(\cdot) : \mathbb{R} \to (-\pi/2, \pi/2).$$

Sketch the graphs of these functions and show that

$$\arcsin'(x) = \frac{1}{\sqrt{1 - x^2}}, \quad \arccos'(x) = -\frac{1}{\sqrt{1 - x^2}}, \qquad x \in (-1, 1),$$

and

$$\arctan'(x) = \frac{1}{1 + x^2}, \qquad x \in \mathbb{R}.$$