SAARLAND UNIVERSITY
Department of Mathematics
Prof. Dr. Mark Groves
MSc Jens Horn

Mathematics for Computer Scientists 1, WS 2017/18
Sheet 5

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1. Prove the following results using the pigeon-hole principle.

(a) In every collection of 7 integers there are at least two whose difference is divisible by 6.

(b) Let n be a natural number. In every collection of n^2+1 points $P_1, \ldots, P_{n^2+1}$ in a square of side length n there are at least two points separated by a distance of no more than $\sqrt{2}$.

(c) In every collection of 51 integers between 1 and 100 there are at least two whose sum is 101.

2. Prove that the set of all prime numbers is infinite. [Hint: Modify the proof that the set $\mathbb{N}$ is infinite. You may assume that a natural number $m \geq 2$ is either a prime number or divisible by a prime number.]

3. (a) Prove that the set of all finite subsets of $\mathbb{N}$ is countably infinite. [Hint: Arrange the subsets according to the sum of their elements.]

(b) Let A_1, A_2, A_3, ... be countably infinite sets. Prove that $\bigcup\limits_{i=1}^{\infty} A_i$ is also countably infinite. [Hint: Denote the elements of A_i by $\{a_{i,1}, a_{i,2}, a_{i,3}, a_{i,4}, \ldots\}$ and apply a diagonal argument to count the elements $\{a_{i,j}\}_{i,j=1,2,\ldots}$ of $\bigcup\limits_{i=1}^{\infty} A_i$.]

4. (a) Compute the solution set of the simultaneous equations

$$x \equiv 2 \pmod{3},$$
$$x \equiv 5 \pmod{7},$$
$$x \equiv 8 \pmod{11}$$

by applying the Chinese remainder theorem twice.

(b) What are the last two digits of the number 49^{19}? [Hint: We want to compute the number $49^{19} \pmod{100}$. Note that $100 = 25 \times 4$.]